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The correlation functions in the 32-vertex model

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Abstract. All the correlation functions in the free-fermion 32-vertex model are calculated.

1. Introduction

Recently, new calculational methods (Samuel 1978a, b, 1979a) have been developed to attack statistical mechanics problems. The methods cover a wide range of interesting systems. Among them are the d-dimensional Ising model, the two-dimensional ferroelectric vertex models, dimer models, and polymer systems. The techniques are neither limited to two dimensions nor to solvable systems: they can be applied to unsolvable models via interesting approximation schemes (Samuel 1979a). The idea is quite simple: statistical mechanics systems are written as fermionic field theories. In contrast to previous fermionisations (Green and Hurst 1964, Schultz *et al* 1964, Hurst 1966), this is done directly by using a path integral formulation. Such an integral is over anticommuting variables (Berezin 1966). These anticommuting variables are the new tools. They are very powerful mathematically and generate both exact results and new approximation schemes. In certain cases, the fermionic action is quadratic, in which case the model is exactly solvable. This happens for all models previously solvable via Pfaffian methods (Kasteleyn 1961, 1963, Montroll *et al* 1963, Green and Hurst 1964, Montroll 1964, 1968). In fact, it is known that an integral over a quadratic fermionic action is a Pfaffian (Berezin 1966, Hurst 1966, Samuel 1978a). The anticommuting variables, however, have many advantages over the Pfaffian methods. They are easier to manipulate, technical problems such as minus signs are easier to handle, and computations are more rapid and more simple. More importantly, anticommuting variables can attack unsolvable models (Samuel 1979a) not accessible to Pfaffian methods.

This paper considers the 32-vertex model of Sacco and Wu (1975). They have computed the partition function and discussed its phase structure. This exactly solvable model has a quadratic fermion action. This paper computes all the correlation functions. This has already been done for the two-dimensional Ising model (Samuel 1978b) and the free-fermion eight-vertex model (Samuel 1979a) where anticommuting variables yield a simple set of computational rules. Thus, all correlation functions of these two models are known. The purpose of this short paper is to extend the results to the 32-vertex model.

The technique is simple and applicable to any model solvable by the old Pfaffian methods. First, determine the quadratic action. Second, go to momentum space via

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Fourier transform and calculate the two-point momentum space anticommuting variable correlation functions. Then return to coordinate space and determine the two-point coordinate space anticommuting variable correlation functions. Finally, relate physical correlations to anticommuting variable ones and use the general formulae for free fermionic field theories.

The anticommuting variables are powerful mathematically but have little physical interpretation. Therefore, the new methods are useful computationally but only indirectly useful for physical considerations. Physical intuition and other input are necessary to determine the phase structure. For this reason this paper emphasises the mathematical aspects of the 32-vertex model. The physical properties have been discussed by Sacco and Wu (1975), to which the reader is referred.

The 32-vertex model places bonds on the edges of a triangular lattice. An even number of bonds (or solid lines) must be incident to each site (see Sacco and Wu (1975) for a more complete description of the model). This is equivalent to drawing closed polygons on the triangular lattice such that polygonal sides cannot overlap. Polygons may intersect themselves or other polygons at a vertex. The anticommuting variable action draws these polygons. Because sides cannot overlap anticommuting variables are ideal: sides constructed out of them cannot overlap because the square of an anticommuting variable is zero (a sort of Pauli exclusion principle). The anticommuting variable action is discussed in the Appendix.

There are 32 possible configurations at a vertex. These are displayed in figure 1 of Sacco and Wu (1975). By appropriately replacing dotted and solid lines by arrowed lines a ferroelectric model is obtained. The general 32-vertex model assigns arbitrary weights (Boltzmann factors) to the 32 possibilities. Sacco and Wu denote the weights by $f_0, \bar{f}_0, f_{ij}$ and $\bar{f}_{ij}$ ($i<j$ and i and j range from 1 to 6) (see figure 1 of Sacco and Wu (1975)). The partition function is a sum over all possible configurations weighted by a product of the vertex Boltzmann factors. The most general vertex model is not yet exactly solvable. Constraints must be imposed. These are called the free-fermion constraints. They guarantee that the anticommuting variable action involves only quadratic terms and not quartic ones. The constraints are

$$\begin{aligned} &\bar{f}_{ij} = (-1)^{i+j} \sum_{\substack{l<m \\ n<p}} \epsilon_{ijlmnp} f_{lm} f_{np} \qquad (i<j) \\ &f_0 \bar{f}_0 = f_{12}\bar{f}_{12} - f_{13}\bar{f}_{13} + f_{14}\bar{f}_{14} - f_{15}\bar{f}_{15} + f_{16}\bar{f}_{16}. \end{aligned} \tag{1.1}$$

Hence 16 vertex weights ($\bar{f}_0$ and $\bar{f}_{ij}$ for $i<j$) are determined in terms of the other 16 (f_0 and f_{ij} for $i<j$). In addition, one may multiply the partition function by a constant to fix one of the weights. We choose

$$f_0 = -1. \tag{1.2}$$

Although the model of Sacco and Wu (1975) is not the most general 32-vertex model, it still has 15 free parameters. It is the most general easily solvable model on a triangular lattice.

This paper adopts the conventions and notation of Sacco and Wu (1975). Study of this reference is recommended before reading this paper. Not only will this familiarise the reader with the notation, but it will provide a framework for the physics, phase structure and nature of the 32-vertex model.

For convenience distort the triangular lattice into the lattice of figure 1. Now vertices lie on integer lattice sites and their location can be specified by two Cartesian coordinates (α, β).

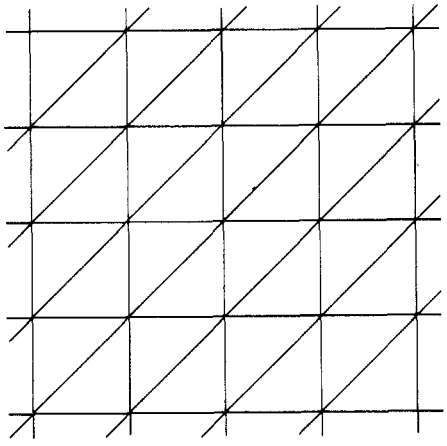

Figure 1. The lattice for the 32-vertex model.

The weight of a configuration will be used to denote the type of vertex configuration. Saying that configuration f_0 (the first diagram of figure 1 of Sacco and Wu (1975)) occurs at (α, β) means that all lines incident to this vertex are dotted.

Section 2 contains the results: all the correlation functions are given in equation (2.9). The partition function is given in equation (2.5) and agrees with Sacco and Wu (1975). Section 2 contains no detailed computations (for details see the Appendix). Instead, only definitions needed to understand equation (2.9) are presented. For this reason, § 2 can be read without knowledge of the anticommuting variable methods. The Appendix discusses the anticommuting variable action, the relation to the 128-vertex model (Samuel 1979b), the physical vertex operators and the computational details. The method is quite general and applicable to any model solvable by Pfaffian methods. Section 3 contains a brief summary.

2. The correlation functions

The main result of this paper, the calculation of the physical correlation functions in the 32-vertex model, is presented in this section. Let

$$P = P^{(c_1)(c_2)\ldots(c_m)}_{\alpha_1\beta_1\alpha_2\beta_2\ldots\alpha_m\beta_m} \tag{2.1}$$

be the probability that configurations (c_1) through (c_m) appear at lattice sites (α_1, β_1) through (α_m, β_m). The abbreviation, P, will be used in lieu of $P^{(c_1)\ldots(c_m)}_{\alpha_1\beta_1\ldots\alpha_m\beta_m}$ with the superscripts $(c_1), \ldots, (c_m)$ and the subscripts $\alpha_1\beta_1, \ldots, \alpha_m\beta_m$ understood. Each (c_s) stands for $\bar{f}_0$, an $\bar{f}_{ij}$, an f_{ij} or f_0. The quantities in equation (2.1) are the physical correlation functions. They are all calculated (in equation (2.9)) in terms of a Pfaffian of a $6m \times 6m$ matrix.

Define a matrix, **M**, by

$$M_{ij} \equiv (-1)^{i+j} f_{ij} + F_{ij}(p_x, p_y). \tag{2.2}$$

The f_{ij} are the same quantities as in Sacco and Wu (1975) for $i<j$. For $i>j$ define $f_{ij} = -f_{ji}$. The non-zero F_{ij}'s are

$$\begin{aligned} F_{14}(p_x) &\equiv -\exp(\mathrm{i}p_x) \equiv -F_{41}(-p_x) \\ F_{25}(p_x, p_y) &\equiv -\exp(\mathrm{i}p_x + \mathrm{i}p_y) \equiv -F_{52}(-p_x, -p_y) \\ F_{36}(p_y) &\equiv -\exp(\mathrm{i}p_y) \equiv -F_{63}(-p_x, -p_y). \end{aligned} \tag{2.3}$$

With these definitions **M** is anti-Hermitian matrix. Let

$$D(p_x, p_y) = \det M(p_x, p_y). \tag{2.4}$$

The determinant, D, has been computed by Sacco and Wu (1975) and enters in the free energy per unit volume:

$$\frac{1}{\text{volume}} \ln Z = \tfrac{1}{2} \int_{-\pi}^{\pi} \frac{\mathrm{d}p_x}{2\pi} \frac{\mathrm{d}p_y}{2\pi} \ln D(p_x, p_y). \tag{2.5}$$

The thermodynamic properties of the 32-vertex model were extracted by Sacco and Wu using equation (2.5).

Define a set of quantities

$$\begin{aligned} a_{ij}(p_x, p_y) &= -a_{ji}(p_x, p_y) \equiv (M^{-1})_{ij} \\ &= \frac{\text{cofactor of } M_{ij}(p_x, p_y)}{D(p_x, p_y)} \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \langle \eta^i_{\alpha\beta} \eta^j_{\alpha'\beta'} \rangle &\equiv -\langle \eta^j_{\alpha'\beta'} \eta^i_{\alpha\beta} \rangle \\ &\equiv \int_{-\pi}^{\pi} \frac{\mathrm{d}p_x}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}p_y}{2\pi} \exp[\mathrm{i}p_x(\alpha - \alpha') + \mathrm{i}p_y(\beta - \beta')] a_{ij}(p_x, p_y). \end{aligned} \tag{2.7}$$

The superscripts i and j run from 1 to 6. The subscripts (α, β) refer to lattice locations as in equation (2.1). There are thus six η's per site. For notational convenience use the abbreviations

$$\eta_{6s-6+l} \equiv \eta^l_{\alpha_s \beta_s} \tag{2.8}$$

for $s = 1$ to m and $l = 1$ to 6. According to equation (2.8) the first 6 η's at (α_1, β_1) have been relabelled $\eta_1, \eta_2, \ldots, \eta_6$; The next 6 η's at (α_2, β_2) have been relabelled $\eta_7, \eta_8, \ldots, \eta_{12}$; and so on.

The configuration probability, P, in equation (2.1) is

$$P = \left(\prod_{s=1}^{m} F^{(c_s)} \right) P_f G, \tag{2.9}$$

where P_f stands for Pfaffian and

$$G_{ij} \equiv \langle \eta_i \eta_j \rangle + \Delta_{ij} \tag{2.10}$$

for i and j equal to 1 through $6m$. The first term, $\langle \eta_i \eta_j \rangle$, is the quantity defined in

equation (2.7) using the abbreviations in equation (2.8). It remains to define $F^{(c)}$ and Δ_{ij}:

$$F^{f_0} \equiv \bar{f}_0 \equiv -F^{\bar{f}_0}$$
$$F^{\bar{f}_{ij}} \equiv (-1)^{i+j}\bar{f}_{ij}f_{ij} \equiv -F^{f_{ij}} \qquad i<j. \tag{2.11}$$

The Δ_{ij} are more awkward to define. Δ_{ij} is antisymmetric in i and j so that G in equation (2.10) is an antisymmetric matrix. It is useful to group the i and j indices in clusters of six as in equation (2.8). The integers 1, 2, 3, 4, 5 and 6 compose the first cluster (and refer to the first vertex (α_1, β_1) in equation (2.1)); the integers 7, 8, 9, 10, 11 and 12 compose the second cluster (and refer to the second vertex); etc. Δ_{ij} vanishes if i and j are of different clusters:

$$\Delta_{ij} = 0 \qquad \text{if} \qquad \begin{matrix} i = 6s-6+l \\ j = 6s'-6+l' \end{matrix} \qquad s \neq s' \tag{2.12}$$

where l and l' range from 1 to 6. In equation (2.12) i would be a member of the sth cluster, whereas j would be a member of the s'th cluster. It is only necessary to define Δ_{ij} for i and j in the same sth cluster. This depends on the configuration (c_s) associated with this cluster.

Table 1. Summary of Δ_{ij}'s and F's. Δ_{ij} and F depend on the type of vertex configuration. In the first column are the 32 possible configurations; in the second and third columns are the corresponding Δ_{ij} and F. Actually, column two displays Δ_{ij} for the first cluster so that subscripts range from 1 to 6. To obtain the Δ_{ij} for the sth cluster shift i and j by $6s-6$.

Configuration	Δ_{ij}	F
$\bar{f}_0$	all Δ_{ij} are zero	$-\bar{f}_0$
$\bar{f}_{ab}$	$\Delta_{ab} = (-1)^{a+b+1}/f_{ab} \qquad (a<b)$ $\Delta_{ij} = 0 \qquad$ for $\quad i \neq a \quad$ or $\quad j \neq b$	$(-1)^{a+b}f_{ab}\bar{f}_{ab}$
f_{ab}	$\Delta_{ln} = (-1)^{c+d}\epsilon_{ablncd}f_{cd}/\bar{f}_{ab} \qquad (a<b,\ (l<n,\ c<d)$ $\Delta_{ln} = 0 \qquad$ for $\qquad l = a,\ n = b$	$(-1)^{a+b+1}f_{ab}\bar{f}_{ab}$
f_0	$\Delta_{ln} = -\bar{f}_{ln}/\bar{f}_0$	$\bar{f}_0$

If $(c_s) = \bar{f}_0$,

$$\Delta_{ij} = 0 \tag{2.13}$$

for all i and j in the sth cluster, i.e., for $i, j = 6s-6+l$ $(l = 1, 2, \ldots, 6)$.

If $(c_s) = \bar{f}_{ab}$ $(a<b)$,

$$\Delta_{ij} = \frac{(-1)^{a+b+1}}{f_{ab}} \qquad \text{for} \qquad \begin{matrix} i = 6s-6+a \\ j = 6s-6+b. \end{matrix} \tag{2.14}$$

$\Delta_{ij} = 0$ for all other i, j in the sth cluster, for $i<j$.

If $(c_s) = f_{ab}$ $(a < b)$,

$$\Delta_{ij} = (-1)^{c+d} \epsilon_{ablncd} f_{cd} / \bar{f}_{ab} \quad \text{for} \quad \begin{matrix} i = 6s-6+l & c<d \\ j = 6s-6+n & l<n \end{matrix}$$
$$\Delta_{ij} = 0 \quad \text{for} \quad \begin{matrix} i = 6s-6+a \text{ or } 6s-6+b \\ j = 6s-6+a \text{ or } 6s-6+b, \end{matrix} \tag{2.15}$$

and $i < j$.

If $(c_s) = f_0$,

$$\Delta_{ij} = -\bar{f}_{ln} / \bar{f}_o \quad \text{for} \quad \begin{matrix} i = 6s-6+l & i<j \\ j = 6s-6+n & l<n. \end{matrix} \tag{2.16}$$

For $i > j$ $\Delta_{ij} \equiv -\Delta_{ji}$. In equation (2.15) ϵ_{ablncd} is the completely antisymmetric tensor with $\epsilon_{123456} \equiv 1$.

3. Conclusion

This paper has calculated all the correlation functions in the free-fermion 32-vertex model by employing the newly developed anticommuting variable techniques. The emphasis was on the mathematical properties. Analysis of the correlation functions (equation (2.9)) near the critical regions awaits further study.

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Appendix. Grassmann integral formulation

This section is intended for those familiar with the anticommuting variable techniques (Samuel 1978a, b, 1979a).

The partition function for the 32-vertex model can be expressed as

$$Z_{32\text{-vertex}} = \int d\eta \exp(A_{32\text{-vertex}}) \tag{A.1}$$

where

$$A_{32\text{-vertex}} = \tfrac{1}{2} \sum_{\alpha\beta} \sum_{ij} (-1)^{i+j} f_{ij} \eta^i_{\alpha\beta} \eta^j_{\alpha\beta} + \sum_{\alpha\beta} (\eta^4_{\alpha\beta} \eta^1_{\alpha+1\beta} + \eta^5_{\alpha\beta} \eta^2_{\alpha+1\beta+1} + \eta^6_{\alpha\beta} \eta^3_{\alpha\beta+1}). \tag{A.2}$$

There are six anticommuting variables per (α, β) site: $\eta^1_{\alpha\beta}, \eta^2_{\alpha\beta}, \ldots, \eta^6_{\alpha\beta}$. The pair (α, β) represents the Cartesian coordinates of the variable on the lattice of figure 1. Equation (A.1) is an integral over all anticommuting variables at all sites and equation

(A.2) is the anticommuting variable action. Because the action is quadratic the model is solvable. This action draws closed polygons on the lattice of figure 1 (it should be noted that in regard to the references (Samuel 1978a, 1979b) equation (A.1) draws the polygons constructed out of the dotted (as opposed to solid) lines of figure 1 of Sacco and Wu (1975)). The first term in equation (A.2) contains the corners and the monomers, while the second term draws the sides of the polygons. It is easy to verify that equations (A.1) and (A.2) produces the weights of the vertex configurations of figure 1 of Sacco and Wu (1975). The only complication is the overall sign due to anticommuting variable reordering. This is a subcase of the 128-vertex model (Samuel 1979b) and overall signs were determined in appendix A of Samuel (1979b). With regard to this reference the superscripts on the η's which determine the type of variable correspond to

$$\begin{aligned} 1 &\leftrightarrow \mathrm{h} & 4 &\leftrightarrow \mathrm{h}^\dagger \\ 2 &\leftrightarrow \mathrm{d} & 6 &\leftrightarrow \mathrm{v}^\dagger \\ 3 &\leftrightarrow \mathrm{v} & 5 &\leftrightarrow \mathrm{d}^\dagger \end{aligned} \tag{A.3}$$

where h, d and v stand for horizontal, vertical and diagonal (see Samuel 1979b), the three directions in figure 1.

To solve the 32-vertex model, go to momentum space by writing

$$\eta^i_{\alpha\beta} = \frac{1}{\sqrt{V}} \sum_{p_x,\, p_y} \exp(\mathrm{i}\alpha p_x + \mathrm{i}\beta p_y) \tilde{a}^i(p_x, p_y) \tag{A.4}$$

where V is the number of sites, p_x and p_y are a discrete set of lattice momenta, and $\tilde{a}^i(p_x, p_y)$ are anticommuting variables in momentum space. It is useful to redefine

$$\begin{aligned} \tilde{a}^i(-p_x, -p_y) &= a^{i\dagger}(p_x, p_y) \\ \tilde{a}^i(p_x, p_y) &= a^i(p_x, p_y) \end{aligned} \qquad \text{for} \quad p_x > 0, \tag{A.5}$$

in which case the action in momentum space becomes

$$A_{32\text{-vertex}} = \sum_{p_x > 0} \sum_{p_y} \sum_{ij} a^i(p_x, p_y) M_{ij}(p_x, p_y) a^{i\dagger}(p_x, p_y), \tag{A.6}$$

where $M(p_x, p_y)$ is the matrix defined in equation (2.2). It immediately follows that the partition function in the thermodynamic limit is given by equations (2.4) and (2.5). The anticommuting variable technique has solved this model in just a few lines of algebra.

Another consequence of equation (A.6) is that the momentum space correlation functions, $\langle a^i(p_x, p_y)\, a^{i\dagger}(p_x, p_y)\rangle$, are given by $(M(p_x, p_y))^{-1}_{ij}$, i.e., the quantities in equation (2.6). A calculation in coordinate space yields equation (2.7). Equations (2.7) are the anticommuting variable correlations. These quantities are useful to know because operators which produce vertex configurations can be constructed out of the anticommuting variables:

$$\begin{aligned} O^{\bar{f}_0}_{\alpha\beta} &= -\bar{f}_0 \eta^1_{\alpha\beta} \eta^2_{\alpha\beta} \eta^3_{\alpha\beta} \eta^4_{\alpha\beta} \eta^5_{\alpha\beta} \eta^6_{\alpha\beta} \\ O^{\bar{f}_{ij}}_{\alpha\beta} &= -\bar{f}_{ij} \epsilon^{ijl_1l_2l_3l_4} \eta^{l_1}_{\alpha\beta} \eta^{l_2}_{\alpha\beta} \eta^{l_3}_{\alpha\beta} \eta^{l_4}_{\alpha\beta} (1 - (-1)^{i+j} f_{ij} \eta^i_{\alpha\beta} \eta^i_{\alpha\beta}) \\ O^{f_{ij}}_{\alpha\beta} &= (-1)^{i+j} f_{ij} \eta^i_{\alpha\beta} \eta^j_{\alpha\beta} \prod_{\substack{l<n \\ l\neq i \text{ or } j \\ n\neq \text{ or } j}} (1 - (-1)^{l+n} f_{ln} \eta^l_{\alpha\beta} \eta^n_{\alpha\beta}) \end{aligned} \tag{A.7}$$

$$O^{f_o}_{\alpha\beta} = \prod_{l<n} (1-(-1)^{l+n} f_{ln} \eta^{l}_{\alpha\beta} \eta^{n}_{\alpha\beta}).$$

When an operator $O^{(c)}_{\alpha\beta}$ is inserted in the integrand of equation (A.1), the configuration (c) must occur at the (α, β) site. In other words, the probability defined in equation (2.1) is

$$P = \langle O^{(c_1)}_{\alpha_1\beta_1} O^{(c_2)}_{\alpha_2\beta_2} \ldots O^{(c_m)}_{\alpha_m\beta_m} \rangle \tag{A.8}$$

where the expectation is taken with respect to equation (A.1). Equation (2.9) as well as the definitions of $F^{(c)}$ and Δ_{ij} follow from equations (A.7) and (A.8).

References

Berezin F A 1966 *The Method of Second Quantization* (New York: Academic)
Green H S and Hurst C A 1964 *Order–Disorder Phenomena* (New York: Interscience)
Hurst C A 1966 *J. Math. Phys.* **7** 305–10
Kasteleyn P W 1961 *Physica* **27** 1209–25
—— 1963 *J. Math. Phys.* **4** 287–93
Montroll E W 1964 *Applied Combinational Mathematics* ed E F Beckenbach (New York: Wiley)
—— 1968 *Brandeis University Summer Institute in Theoretical Physics* 1966 ed M Chretien, E P Gross and S Deser (New York: Gordon and Breach)
Montroll E W, Potts R B and Ward J C 1963 *J. Math. Phys.* **4** 308–22
Sacco J E and Wu F Y 1975 *J. Phys. A: Math. Gen.* **8** 1780–7
Samuel S A 1978a *LBL Preprint* 8217 (1980 *J. Math. Phys.* to be published)
—— 1978b *LBL Preprint* 8300 (1980 *J. Math. Phys.* to be published)
—— 1979a *LBL Preprint* 9347 (1980 *J. Math. Phys.* to be published)
—— 1979b *LBL Preprint* 8800 (1981 *J. Phys. A: Math. Gen.* **14** 191)
Schultz T D, Mattis D C and Lieb E H 1964 *Rev. Mod. Phys.* **36** 856–71